7.1 nth Roots and Rational Exponents

**What you should learn**

**GOAL 1** Evaluate nth roots of real numbers using both radical notation and rational exponent notation.

**GOAL 2** Use nth roots to solve real-life problems, such as finding the total mass of a spacecraft that can be sent to Mars in Example 5.

**Why you should learn it**

To solve real-life problems, such as finding the number of reptile and amphibian species that Puerto Rico can support in Ex. 67.

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**nith Roots and Rational Exponents**

**GOAL 1** EVALUATING nth ROOTS

You can extend the concept of a square root to other types of roots. For instance, 2 is a cube root of 8 because \(2^3 = 8\), and 3 is a fourth root of 81 because \(3^4 = 81\). In general, for an integer \(n\) greater than 1, if \(b^n = a\), then \(b\) is an \(n\)th root of \(a\).

An \(n\)th root of \(a\) is written as \(\sqrt[n]{a}\), where \(n\) is the index of the radical.

You can also write an \(n\)th root of \(a\) as a power of \(a\). For the particular case of a square root, suppose that \(\sqrt{a} = a^{1/2}\). Then you can determine a value for \(k\) as follows:

\[
\sqrt{a} \cdot \sqrt{a} = a \quad \text{Definition of square root}
\]
\[
a^k \cdot a^k = a \quad \text{Substitute } a^k \text{ for } \sqrt{a}.
\]
\[
a^{2k} = a^1 \quad \text{Product of powers property}
\]
\[
2k = 1 \quad \text{Set exponents equal when bases are equal.}
\]
\[
k = \frac{1}{2} \quad \text{Solve for } k.
\]

Therefore, you can see that \(\sqrt{a} = a^{1/2}\). In a similar way you can show that \(\sqrt[3]{a} = a^{1/3}\) and \(\sqrt[4]{a} = a^{1/4}\). In general, \(\sqrt[n]{a} = a^{1/n}\) for any integer \(n\) greater than 1.

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**REAL nth ROOTS**

Let \(n\) be an integer greater than 1 and let \(a\) be a real number.

- If \(n\) is odd, then \(a\) has one real \(n\)th root: \(\sqrt[n]{a} = a^{1/n}\)
- If \(n\) is even and \(a > 0\), then \(a\) has two real \(n\)th roots: \(\pm \sqrt[n]{a} = \pm a^{1/n}\)
- If \(n\) is even and \(a = 0\), then \(a\) has one \(n\)th root: \(\sqrt[0]{0} = 0^{1/n} = 0\)
- If \(n\) is even and \(a < 0\), then \(a\) has no real \(n\)th roots.

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**EXAMPLE 1** Finding nth Roots

Find the indicated real \(n\)th root(s) of \(a\).

a. \(n = 3, a = -125\)  

b. \(n = 4, a = 16\)

**SOLUTION**

a. Because \(n = 3\) is odd, \(a = -125\) has one real cube root. Because \((-5)^3 = -125\), you can write:

\[
\sqrt[3]{-125} = -5 \quad \text{or} \quad (-125)^{1/3} = -5
\]

b. Because \(n = 4\) is even and \(a = 16 > 0\), 16 has two real fourth roots. Because \(2^4 = 16\) and \((-2)^4 = 16\), you can write:

\[
\pm \sqrt[4]{16} = \pm 2 \quad \text{or} \quad \pm 16^{1/4} = \pm 2
\]
A rational exponent does not have to be of the form \( \frac{1}{n} \) where \( n \) is an integer greater than 1. Other rational numbers such as \( \frac{3}{2} \) and \( -\frac{1}{2} \) can also be used as exponents.

### RATIONAL EXPONENTS

Let \( a^{1/n} \) be an \( n \)th root of \( a \), and let \( m \) be a positive integer.

- \( a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m \)
- \( a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(a^{1/n})^m} = \frac{1}{(\sqrt[n]{a})^m}, \ a \neq 0 \)

### EXAMPLE 2 Evaluating Expressions with Rational Exponents

- \( 9^{3/2} = (\sqrt{9})^3 = 3^3 = 27 \quad \text{Using radical notation} \)
- \( 9^{3/2} = (9^{1/2})^3 = 3^3 = 27 \quad \text{Using rational exponent notation} \)

- \( 32^{-2/5} = \frac{1}{32^{2/5}} = \frac{1}{(\sqrt[5]{32})^2} = \frac{1}{2^2} = \frac{1}{4} \quad \text{Using radical notation} \)
- \( 32^{-2/5} = \frac{1}{32^{2/5}} = \frac{1}{(32^{1/5})^2} = \frac{1}{2^2} = \frac{1}{4} \quad \text{Using rational exponent notation} \)

When using a graphing calculator to approximate an \( n \)th root, you may have to rewrite the \( n \)th root using a rational exponent. Then use the calculator’s power key.

### EXAMPLE 3 Approximating a Root with a Calculator

Use a graphing calculator to approximate \((\sqrt[3]{5})^3\).

**SOLUTION** First rewrite \((\sqrt[3]{5})^3\) as \(5^{3/4}\). Then enter the following:

Keystrokes: \( 5 \) \( \div \) \( 3 \) \( \div \) \( 4 \) \( \) \( \) \( \) \( \) \( \text{ENTER} \)

Display: \( 3.343701525 \)

\((\sqrt[3]{5})^3 \approx 3.34\)

To solve simple equations involving \( x^n \), isolate the power and then take the \( n \)th root of each side.

### EXAMPLE 4 Solving Equations Using nth Roots

- \( 2x^4 = 162 \) \quad \( x^4 = 81 \) \quad \( x = \pm \sqrt[4]{81} \) \quad \( x = \pm 3 \)
- \( (x - 2)^3 = 10 \) \quad \( x - 2 = \sqrt[3]{10} \) \quad \( x = \sqrt[3]{10} + 2 \) \quad \( x \approx 4.15 \)
GOAL 2 USING NTH ROOTS IN REAL LIFE

EXAMPLE 5 Evaluating a Model with nth Roots

The total mass $M$ (in kilograms) of a spacecraft that can be propelled by a magnetic sail is, in theory, given by

$$M = \frac{0.015m^2}{fd^{4/3}}$$

where $m$ is the mass (in kilograms) of the magnetic sail, $f$ is the drag force (in newtons) of the spacecraft, and $d$ is the distance (in astronomical units) to the sun. Find the total mass of a spacecraft that can be sent to Mars using $m = 5000$ kg, $f = 4.52$ N, and $d = 1.52$ AU. **Source: Journal of Spacecraft and Rockets**

**SOLUTION**

$$M = \frac{0.015(5000)^2}{4.52(1.52)^{4/3}}$$

$$= \frac{0.015(25000000)}{4.52 \cdot 2.59}$$

$$= \frac{37500000}{11.82}$$

$$\approx 47500$$

The spacecraft can have a total mass of about 47,500 kilograms. (For comparison, the liftoff weight for a space shuttle is usually about 2,040,000 kilograms.)

EXAMPLE 6 Solving an Equation Using an nth Root

**NAUTICAL SCIENCE** The *Olympias* is a reconstruction of a trireme, a type of Greek galley ship used over 2000 years ago. The power $P$ (in kilowatts) needed to propel the *Olympias* at a desired speed $s$ (in knots) can be modeled by this equation:

$$P = 0.0289s^3$$

A volunteer crew of the *Olympias* was able to generate a maximum power of about 10.5 kilowatts. What was their greatest speed? **Source: Scientific American**

**SOLUTION**

$$P = 0.0289s^3$$

$$10.5 = 0.0289s^3$$

$$363 = s^3$$

$$\sqrt[3]{363} = s$$

$$7 \approx s$$

The greatest speed attained by the *Olympias* was approximately 7 knots (about 8 miles per hour).
GUIDED PRACTICE

Vocabulary Check ✓
Concept Check ✓

1. What is the index of a radical?
2. LOGICAL REASONING Let \( n \) be an integer greater than 1. Tell whether the given statement is always true, sometimes true, or never true. Explain.
   a. If \( x^n = a \), then \( x = \sqrt[n]{a} \).
   b. \( a^{1/n} = 1 \)

3. Try to evaluate the expressions \(-\sqrt[4]{625}\) and \(\sqrt[4]{625}\). Explain the difference in your results.

Evaluate the expression.
4. \( \sqrt[8]{1} \)
5. \(-\left(49^{1/2}\right)\)
6. \(\left(\sqrt{-8}\right)^5\)
7. \(3125^{2/5}\)

Solve the equation.
8. \(x^3 = 125\)
9. \(3x^5 = -3\)
10. \((x + 4)^2 = 0\)
11. \(x^4 - 7 = 993\)

12. **SHOT PUT** The shot (a metal sphere) used in men’s shot put has a volume of about 905 cubic centimeters. Find the radius of the shot. (Hint: Use the formula \(V = \frac{4}{3}\pi r^3\) for the volume of a sphere.)

PRACTICE AND APPLICATIONS

USING RATIONAL EXPONENT NOTATION Rewrite the expression using rational exponent notation.
13. \(\sqrt[14]{1}\)
14. \(\sqrt[11]{1}\)
15. \(\left(\sqrt[5]{5}\right)^2\)
16. \(\left(\sqrt[16]{16}\right)^5\)
17. \(\left(\sqrt[2]{2}\right)^{11}\)

USING RADICAL NOTATION Rewrite the expression using radical notation.
18. \(6^{1/3}\)
19. \(7^{1/4}\)
20. \(10^{3/7}\)
21. \(5^{2/5}\)
22. \(8^{7/4}\)

FINDING \(n\)TH ROOTS Find the indicated real \(n\)th root(s) of \(a\).
23. \(n = 2, a = 100\)
24. \(n = 4, a = 0\)
25. \(n = 3, a = -8\)
26. \(n = 7, a = 128\)
27. \(n = 6, a = -1\)
28. \(n = 5, a = 0\)

EVALUATING EXPRESSIONS Evaluate the expression without using a calculator.
29. \(\sqrt[64]{64}\)
30. \(-\sqrt[1000]{1000}\)
31. \(-\sqrt[64]{64}\)
32. \(4^{-1/2}\)
33. \(1^{1/3}\)
34. \(-\left(256^{1/4}\right)\)
35. \(\left(\sqrt[16]{16}\right)^2\)
36. \(\left(\sqrt[27]{27}\right)^{-4}\)
37. \(\left(\sqrt[0]{0}\right)^3\)
38. \(-\left(25^{-3/2}\right)\)
39. \(32^{4/5}\)
40. \(-\left(125^{-2/3}\right)\)

APPROXIMATING ROOTS Evaluate the expression using a calculator. Round the result to two decimal places when appropriate.
41. \(\sqrt{-16,807}\)
42. \(\sqrt[112]{112}\)
43. \(\sqrt[5]{65,536}\)
44. \(4^{110}\)
45. \(10^{-1/4}\)
46. \(-\left(1331^{1/3}\right)\)
47. \(\left(\sqrt[112]{112}\right)^{-4}\)
48. \(\left(\sqrt{-280}\right)^3\)
49. \(\left(\sqrt[6]{6}\right)^2\)
50. \((-190)^{-4/5}\)
51. \(26^{-3/4}\)
52. \(522^{2/7}\)
SOLVING EQUATIONS  Solve the equation. Round your answer to two decimal places when appropriate.

53. \( x^5 = 243 \)  
54. \( 6x^3 = -1296 \)  
55. \( x^6 + 10 = 10 \)

56. \( (x - 4)^4 = 81 \)  
57. \( -x^7 = 40 \)  
58. \( -12x^4 = -48 \)

59. \( (x + 12)^3 = 21 \)  
60. \( x^3 - 14 = 22 \)  
61. \( x^8 - 25 = -10 \)

62. **BIOLOGY CONNECTION**  For mammals, the lung volume \( V \) (in milliliters) can be modeled by \( V = 170m^{4/5} \) where \( m \) is the body mass (in kilograms). Find the lung volume of each mammal in the table shown.

<table>
<thead>
<tr>
<th>Mammal</th>
<th>Body mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banded mongoose</td>
<td>1.14</td>
</tr>
<tr>
<td>Camel</td>
<td>229</td>
</tr>
<tr>
<td>Horse</td>
<td>510</td>
</tr>
<tr>
<td>Swiss cow</td>
<td>700</td>
</tr>
</tbody>
</table>

Source: *Respiration Physiology*

63. **SPILLWAY OF A DAM**  A dam’s spillway capacity is an indication of how the dam will perform under certain flood conditions. The spillway capacity \( q \) (in cubic feet per second) of a dam can be calculated using the formula \( q = ch^{3/2} \) where \( c \) is the discharge coefficient, \( l \) is the length (in feet) of the spillway, and \( h \) is the height (in feet) of the water on the spillway. A dam with a spillway 40 feet long, 5 feet deep, and 5 feet wide has a discharge coefficient of 2.79. What is the dam’s maximum spillway capacity?

Source: *Standard Handbook for Civil Engineers*

64. **INFLATION**  If the price of an item increases from \( p_1 \) to \( p_2 \) over a period of \( n \) years, the annual rate of inflation \( i \) (expressed as a decimal) can be modeled by \( i = \left( \frac{p_2}{p_1} \right)^{1/n} - 1 \). In 1940 the average value of a home was $2900. In 1990 the average value was $79,100. What was the rate of inflation for a home?

Source: Bureau of the Census

65. **GEOMETRY CONNECTION**  The formula for the volume \( V \) of a regular dodecahedron (a solid with 12 regular pentagons as faces) is \( V = 7.66a^3 \) where \( a \) is the length of an edge of the dodecahedron. Find the length of an edge of a regular dodecahedron that has a volume of 30 cubic feet. Round your answer to two decimal places.

66. **GEOMETRY CONNECTION**  The formula for the volume \( V \) of a regular icosahedron (a solid with 20 congruent equilateral triangles as faces) is \( V = 2.18a^3 \) where \( a \) is the length of an edge of the icosahedron. Find the length of an edge of a regular icosahedron that has a volume of 21 cubic centimeters. Round your answer to two decimal places.

67. **ISLAND SPECIES**  Philip Darlington discovered a rule of thumb that relates an island’s land area \( A \) (in square miles) to the number \( s \) of reptile and amphibian species the island can support by the model \( A = 0.0779s^3 \). The area of Puerto Rico is roughly 4000 square miles. About how many reptile and amphibian species can it support?

Source: *The Song of the Dodo: Island Biogeography in an Age of Extinctions*
68. **MULTI-STEP PROBLEM** A board foot is a unit for measuring wood. One board foot has a volume of 144 cubic inches. The Doyle log rule, given by \( b = \left( \frac{r}{2} \right)^2 \), is a formula for approximating the number \( b \) of board feet in a log with length \( l \) (in feet) and radius \( r \) (in inches). The total volume \( V \) (in cubic inches) of wood in the main trunk of a Douglas fir can be modeled by \( V = 250r^3 \) where \( r \) is the radius of the trunk at the base of the tree. Suppose you need 5000 board feet from a 20 foot Douglas fir log.

a. What volume of wood do you need?

b. What is the radius of a log that will meet your needs?

c. What is the total volume of wood in the main trunk of a Douglas fir tree that will meet your needs?

d. If you find a suitable tree, what fraction of the tree would you actually use?

e. **Writing** How does your answer to part (d) change if you instead need only 2500 board feet?

69. **VISUAL THINKING** Copy the table. Give the number of \( n \)th roots of \( a \) for each category.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( n ) even</th>
<th>( n ) odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a &lt; 0 )</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( a = 0 )</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>( a &gt; 0 )</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

70. The graph of \( y = x^n \) where \( n \) is even is shown in red. Explain how the graph justifies the table for \( n \) even.

71. Draw a similar graph to justify the table for \( n \) odd.

**MIXED REVIEW**

**SOLVING SYSTEMS** Use Cramer’s rule to solve the linear system. (Review 4.3)

72. \( x + 4y = 12 \)
\( 2x + 5y = 18 \)
73. \( x - 2y = 11 \)
\( 2x + 5y = -14 \)
74. \( 2x - 4y = 7 \)
\( -x + y = 1 \)
75. \( -3x + 2y = -9 \)
\( x - 4y = 2 \)
76. \( -x - 8y = 10 \)
\( 10x + y = 1 \)
77. \( -x - y = 0 \)
\( 5x - 6y = 13 \)

**SIMPLIFYING EXPRESSIONS** Simplify the expression. Tell which properties of exponents you used. (Review 6.1 for 7.2)

78. \( x^4 \cdot x^{-2} \)
82. \( \frac{x^3}{x^{-4}} \)
79. \( (x^{-3})^5 \)
83. \( \left( \frac{x^{-2}}{y} \right)^2 \)
80. \( (2xy)^{-3} \)
84. \( \frac{7x^3y^8}{14xy^{-2}} \)
81. \( 5x^{-2}y^0 \)
85. \( \frac{16xy}{9x^5} \cdot \frac{9x^6y}{4y} \)

**FINDING ZEROS** Find all the zeros of the polynomial function. (Review 6.7)

86. \( f(x) = x^3 + 9x^3 - 5x^2 - 153x - 140 \)
87. \( f(x) = x^4 + x^3 - 19x^2 + 11x + 30 \)
88. \( f(x) = x^3 - 5x^2 + 16x - 80 \)
89. \( f(x) = x^3 - x^2 + 9x - 9 \)